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Existence and some properties of waves trapped by submerged cylinders

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In this paper the mathematical formulation associated with waves trapped by submerged cylinders is recast as a standard eigenvalue problem. In this way the existence is proven of trapped waves for every frequency Ω, and for arbitrary geometry of the submerged cylinder. At the same time a simple expression for the first eigenvalue and eigenmode, correct in the limits $\Omega \to 0$ or $\Omega \to \infty$, is derived. The expression can be a useful approximation for a structure relatively transparent to the wave action such as, for instance, a semisubmersible platform.

1. Introduction

Wave trapping in an uneven bottom topography has been known for a long time. The first report was by Stokes (1846), who showed the existence of edge waves on a plane beach. This matter has been further studied by Ursell (1952) and, in a penetrating work, trapped waves in shallow water have been analysed by Longuet-Higgins (1967).

In this context the basic physical principle can be easily understood. In shallow water the wave velocity is $(gh)^{\frac{1}{2}}$ and over a submarine ridge, uniform in the longitudinal x-direction, this velocity is $[gh(y)]^{\frac{1}{2}} < (gh)^{\frac{1}{2}}$. A wave propagating over the elevation and oblique to the y-direction tends to bend back towards the elevation as it encounters regions of increasing wave velocity. The mathematical problem here is similar to the determination of bound states in a one-dimensional Schrödinger equation, a point that has been observed by Longuet-Higgins.

One way to consider submerged structures is as if they were an abrupt and localized sea bottom elevation. In this sense the existence of trapped waves over them seems plausible, and this fact was proven by Ursell (1951) for a circular submerged cylinder in the low-frequency limit.†

In the last ten or fifteen years relatively slender and submerged structures have been extensively used in offshore engineering. Relevant examples are the semi-submersible platforms and wave energy extraction devices. Since then, the response of such structures to waves has been experimentally and theoretically analysed under different conditions. In particular, D. V. Evans has recently observed, in the laboratory, that 'at certain frequencies the incident wave train would excite large amplitude wave motions, confined to the immediate vicinity of the cylinder, which would persist after the wavemaker was switched off'. This observation has

† One of the referees has called the author's attention to a paper by Jones (1953), where the existence of trapped waves for symmetric bodies has been demonstrated. More recently, Ursell (1987) re-analysed this problem and arrived at conclusions similar to the ones presented in §3 of this work.

apparently motivated him to analyse trapped waves over submerged cylinders; see McIver & Evans (1985).

The purpose of the present paper is to discuss the existence of trapped waves over submerged cylinders, and some of their properties. However, not only is the phenomenon itself not yet well known, but also the mathematical technique to be used in this work – the weak formulation – has not been very much explored in the water-wave specialized literature. To fill this gap, §2 presents a short discussion, based on an analogy with the waveguide problem, where most of the results of this work are anticipated, and a physical background for the mathematical technique is given. In §3, trapped waves are recast as a standard eigenvalue problem and their existence, together with some relevant properties, is demonstrated. In §4, minimum principles of eigenvalue theory are used to determine lower bounds, and a convenient asymptotic approximation is derived for a body not too close to the free surface. This approximation has been used in §5 to derive a necessary condition for trapped-mode excitation, and it can be useful to analyse qualitatively the behaviour of relatively transparent structures such as, for example, a semisubmersible platform.

In a companion paper (Aranha 1988), the excitation of a trapped mode by the incoming wave and the ensuing nonlinear resonant response of a slender structure will be analysed. In this context one may assess the importance that trapped-mode excitation may have for the analysis of a typical ocean structure.

2. Trapped waves: an analogy with the waveguide problem

In this paper a submerged cylinder, with longitudinal axis coincident with the x-direction, will be considered. All variables will be non-dimensionalized using the length $\tilde{B}$, the frequency $\tilde{\Omega}$ and the acceleration due to gravity g. Non-dimensional variables will be indicated by B, Ω, etc., and the geometric definitions are given in figure 1.

At the frequency Ω and water depth h, the wavenumber $K_0(\Omega)$ is given by the dispersion relation

$$\Omega^2 = K_0(\Omega)\tanh(K_0(\Omega)h). \tag{2.1}$$

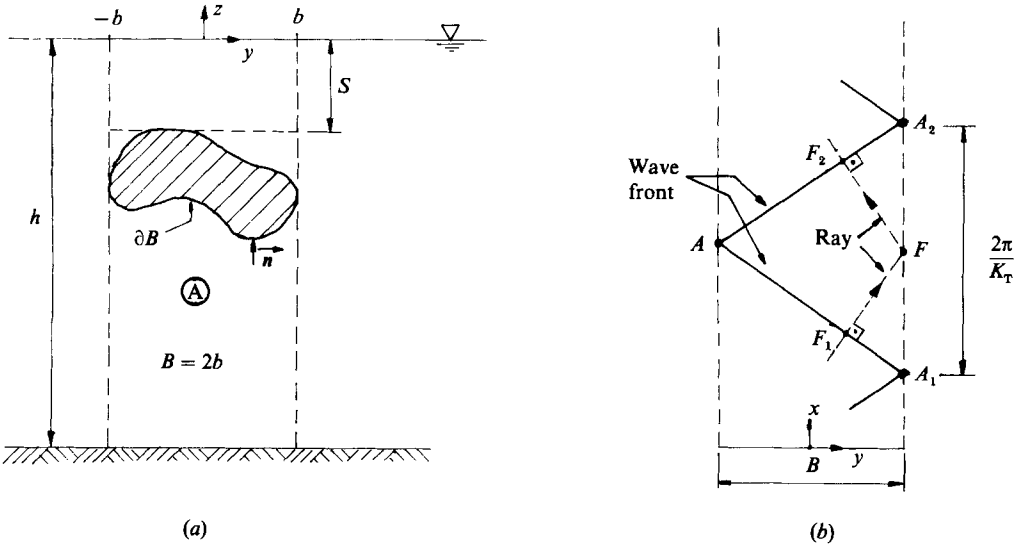
Let S be the minimum distance between the free surface and the submerged body ∂B. If one supposes, for a while, that the body has a rectangular cross-section with sides parallel to the (y,z)-axes then, above the body, the wavenumber is $K_s(\Omega)$, where

$$\Omega^2 = K_s(\Omega)\tanh(K_s(\Omega)S). \tag{2.2}$$

Since certainly $S < h$ then, from (2.1) and (2.2), $K_s(\Omega) > K_0(\Omega)$. In the region $|y| < b$ the wave velocity is $c_s(\Omega) = \Omega/K_s(\Omega)$ and in the region $|y| > b$ it is $c_0(\Omega) = \Omega/K_0(\Omega)$, where $c_s(\Omega) < c_0(\Omega)$. As explained in the introduction, a wave generated in the region $|y| < b$ and oblique to the y-axis, tends to bend back to this region as it encounters an increasing wave velocity field. This wave, confined basically to the region $|y| < b$ and propagating in the x-direction, is called a 'trapped wave'.

A sketch of its wave front A_1AA_2 is shown in figure 1 (b) and it is quite clear that the submerged body works as if it were a waveguide for the trapped wave. From this similarity some results can be inferred by analogy. They will be described here and proven in the next section.

The first is the following: the longitudinal wavenumber $K_T(\Omega)$, shown in figure 1 (b), cannot take an arbitrary value. Indeed, to enforce constructive interference, only a *discrete set* of values can be taken. These discrete values can be associated with the



(*a*) (*b*)

FIGURE 1. (*a*) Geometric definitions for the cross-section. (*b*) Trapped wave guided by the submerged body (plane $z = 0$).

transverse eigenmodes, and they should be determined by means of an *eigenvalue* problem.

From the waveguide analogy it follows also that $K_T(\Omega)$ must have an upper bound, namely $K_T(\Omega) < K_s(\Omega)$. In fact, constructive interference in a waveguide implies that the wave front A_1A, indicated in figure 1(*b*), should have the same phase as the wave front AA_2. In one period the wave front must then cover the distance F_1FF_2, with a local wave velocity of $c_s(\Omega)$. Since A_2 and A_1 have the same phase, the longitudinal wave covers the wavelength A_2A_1 with velocity Ω/K_T, and from the geometric inequality $A_2A_1 > F_1FF_2$ it follows that $\Omega/K_T > c_s(\Omega) = \Omega/K_s$, or $K_T(\Omega) < K_s(\Omega)$.

Details of the waveguide problem can be found in Achenbach (1975), but the trapped wave is not exactly confined to $|y| < b$. It must spread out beyond this region, although it should decay with $|y|$ when $|y| \to \infty$. Its asymptotic behaviour must then be described by

$$\hat{T}(x, y, z, t) \sim A_0^{\pm}\, e^{-\lambda_0|y|} \cosh K_0(z+h)\, e^{i(K_T x - \Omega t)}, \tag{2.3}$$

in the limits $y \to \pm\infty$. Notice that the z-dependent function satisfies both the impermeability condition on the sea bottom $z = -h$ and the free-surface condition $\partial\hat{T}/\partial z = \Omega^2\hat{T}$ on $z = 0$. From the continuity equation and (2.3), one obtains the relation $\lambda_0(\Omega) = (K_T^2(\Omega) - K_0^2(\Omega))^{\frac{1}{2}}$. It follows then that $K_T(\Omega)$ is bounded from below by $K_0(\Omega)$, or

$$K_0(\Omega) < K_T(\Omega) < K_s(\Omega). \tag{2.4}$$

As has been said above, the trapped-mode wavenumber $K_T(\Omega)$ can be determined by a transverse eigenvalue problem. In general the eigenvalues are well separated, and so only a finite number can be located in the finite interval indicated in (2.4). Or, in short, for a given frequency Ω only a *finite number of trapped modes are expected to exist.*†

† This result is essentially the same as the one derived for the bound states of the one-dimensional Schrödinger equation.

If $T(y,z)$ is the transverse eigenmode, the trapped wave can be expressed by

$$\hat{T}(x,y,z,t) = T(y,z)\,\mathrm{e}^{\mathrm{i}(K_{\mathrm{T}}(\Omega)\,x-\Omega t)}. \tag{2.5}$$

In this paper the pair $\{K_{\mathrm{T}}(\Omega);\, T(y,z;\,\Omega)\}$ will be determined by a standard eigenvalue problem obtained, however, by means of a mathematical technique known as 'weak formulation'. To give a clear physical meaning to this technique some results from analytical mechanics will be first recalled here.

In a discrete system, defined by the generalized coordinates $\{q_n(t);\, n = 1,\ldots,N\}$, the Lagrangian is a function of the form $\mathscr{L}(q_n;\dot{q}_n)$. If the system is in free vibration, all generalized coordinates oscillate with the same frequency Ω, and so $q_n(t) = q_{n,0}\,\mathrm{e}^{-\mathrm{i}\Omega t}$. In this case $\mathscr{L} = \mathscr{L}(q_{n,0})$ and the Lagrange equations of motion reduce to $\partial\mathscr{L}/\partial q_{n,0} = 0$. The eigenvalue–eigenmode $\{\Omega;\, q_{n,0}, n = 1,\ldots,N\}$ can be as determined in the following way: Ω is such that there exists a *non-trivial* $\{q_{n,0}\}$ for which the Lagrangian is *stationary*.

In the present work the problem is defined in a *continuum* and the system is described by the field $\hat{T}(x,y,z) = T(y,z)\,\mathrm{e}^{\mathrm{i}(K_{\mathrm{T}}x-\Omega t)}$ instead of by the discrete variables $\{q_{n,0}\}$. If A_∞ is the entire fluid region in the plane (y,z), the non-dimensional kinetic energy is given by

$$E(T) = \frac{1}{2}\iint_{A_\infty} \bar{\nabla}\hat{T}\,\bar{\nabla}\hat{T}^*\,\mathrm{d}A_\infty = \frac{1}{2}\iint_{A_\infty} [(\nabla T)^2 + K_{\mathrm{T}}^2 T^2]\,\mathrm{d}A_\infty. \tag{2.6}$$

In (2.6), the asterisk stands for the complex conjugate, and

$$\nabla = \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}, \quad \bar{\nabla} = \nabla + \boldsymbol{i}\frac{\partial}{\partial x}. \tag{2.7}$$

The potential (gravity) energy is now given by

$$U(T) = \tfrac{1}{2}\Omega^2 \int_{-\infty}^{\infty} T^2(y,0)\,\mathrm{d}y, \tag{2.8}$$

and so the Lagrangian is

$$\mathscr{L}(T) = E(T) - U(T) = \frac{1}{2}\iint_{A_\infty} [(\nabla T)^2 + K_{\mathrm{T}}^2 T^2]\,\mathrm{d}A_\infty - \tfrac{1}{2}\Omega^2 \int_{-\infty}^{\infty} T^2(y,0)\,\mathrm{d}y. \tag{2.9}$$

If K_{T} is supposed given, the eigenvalue problem can now be characterized as: to determine Ω such that there exists a *non-trivial* $T(y,z;\Omega)$ for which the Lagrangian $\mathscr{L}(T)$ is *stationary*. The weak equation to be derived in the next section is just the stationary condition for $\mathscr{L}(T)$.

3. The existence of trapped waves

In the preceding section, some properties concerning trapped waves have been inferred from the waveguide analogy. Furthermore, an eigenvalue problem, based on a Lagrangian formulation, has been proposed as the mathematical framework for the trapped-wave theory. The purpose of the present section is to give a firm mathematical background to these results.

There are two main questions that will be tackled here. The first is to show how, from a classical formulation, one can obtain the stationary condition of (2.9) by means of the so-called 'weak formulation'. The second is to show then that the related eigenvalue problem has a solution. This latter topic is just the existence

theorem for trapped waves, and it will be demonstrated together with the results anticipated in the last section.

A trapped wave is a function of the form (2.5) that satisfies the continuity equation, the free-surface boundary condition $\partial\hat{T}/\partial z = \Omega^2\hat{T}$ at $z = 0$, and the impermeability condition on the body surface ∂B and the sea bottom $z = -h$. Furthermore, it must decay exponentially as $y \to \infty$, and this latter condition requires that $K_T(\Omega) > K_0(\Omega)$; see (2.4). In this context a *trapped mode*† is a pair $\{K = K_T(\Omega);\ \phi = T(y, z; K_T)\}$ for which there exists a non-trivial solution of the following set of equations:

$$\nabla^2\phi - K^2\phi = 0; \quad (y, z) \in A_\infty, \tag{3.1a}$$

$$\frac{\partial\phi}{\partial z}(y, 0) = \Omega^2\phi(y, 0), \tag{3.1b}$$

$$\nabla\phi \cdot \boldsymbol{n}|_{\partial B} = 0, \tag{3.1c}$$

$$\frac{\partial\phi}{\partial z}(y, -h) = 0, \tag{3.1d}$$

$$\frac{\partial\phi}{\partial y}(y, z) \to 0 \quad \text{when } |y| \to \infty; \quad K \geqslant K_0(\Omega). \tag{3.1e}$$

The basic difficulty of this eigenvalue problem is related to the infinite size of the region A_∞, and it can be overcome in the following way: the region A_∞ is divided into a finite part $A(|y| \leqslant b)$, where the body is, and two strips $A^\pm(y \gtrless \pm b)$. In the regions $A^\pm$ the solution of (3.1) will be denoted by $\phi\pm(y, z)$ and these can be expanded in Fourier series using the complete orthonormal set of functions

$$\left(\int_{-h}^{0} f_n(z) f_m(z)\,\mathrm{d}z = \delta_{nm}\right):$$

$$\left.\begin{aligned} f_0(z) &= F_0 \cosh K_0(z+h); \quad \Omega^2 = K_0(\Omega)\tanh K_0(\Omega)\,h, \\ f_n(z) &= F_n \cos\chi_n(z+h); \quad \Omega^2 = -\chi_n(\Omega)\tan\chi_n(\Omega)\,h, \quad n = 1, 2, \ldots. \end{aligned}\right\} \tag{3.2}$$

If $\phi(y, z)$ denotes the potential in the region A, consider the Fourier expansion

$$\left.\begin{aligned} \phi(\pm b, z) &= \sum_{n=0}^{\infty} L_n^\pm(\phi) f_n(z), \\ L_n^\pm(\phi) &= \int_{-h}^{0} \phi(\pm b, z) f_n(z). \end{aligned}\right\} \tag{3.3}$$

The only solution of (3.1) in the regions $A^\pm$, given by (3.3) on the lines $y = \pm b$, can be written as

$$\phi^\pm(y, z) = \sum_{n=0}^{\infty} L_n^\pm(\phi)\, \mathrm{e}^{-\lambda_n(|y|-b)} f_n(z); \quad y \gtrless \pm b, \tag{3.4}$$

with
$$\lambda_0(K) = (K^2 - K_0^2(\Omega))^{\frac{1}{2}}, \quad \lambda_n(K) = (K^2 + \chi_n^2(\Omega))^{\frac{1}{2}}. \tag{3.5}$$

For future reference it is important to observe here the following identity:

$$\sum_{n=0}^{\infty} \lambda_n (L_n^\pm(\phi))^2 = \iint_{A^\pm} [(\nabla\phi^\pm)^2 + K^2(\phi^\pm)^2]\,\mathrm{d}A^\pm - \Omega^2 \int_{F^\pm} (\phi^\pm(y, 0))^2\,\mathrm{d}y, \tag{3.6}$$

† $\hat{T}(x, y, z)$ is the trapped wave, analogous to the guided wave, and $\{K_T(\Omega); T(y, z; K_T)\}$ the trapped mode, analogous to the transverse eigenmode in a waveguide.

where $F^{\pm}$ are the free surfaces of the fluid regions $A^{\pm}$. By definition $\phi(\pm b, z) = \phi^{\pm}(\pm b, z)$, and (3.4) will be the analytic continuation of $\phi(y, z)$ if and only if

$$\frac{\partial \phi}{\partial y}(\pm b, z) = \frac{\partial \phi^{\pm}}{\partial y}(\pm b, z) = \mp \sum_{n=0}^{\infty} \lambda_n(K)\, L_n^{\pm}(\phi) f_n(z). \tag{3.7}$$

In this way the problem has been reduced to the determination of a function $\phi(y, z)$, defined in the finite fluid region A, that satisfies the continuity equation (3.1a) and the boundary conditions (3.1b–d), and (3.7) on the boundary of A. Since interest is centred on a non-trivial solution (eigenvalue problem), it seems natural to look for an energy expression associated with this differential operator, and the first task is to restrict $\phi(y, z)$ to the set of functions with finite energy, namely $\phi(y, z) \in W_2^{(1)}(A)$, where $W_2^{(1)}(A)$ is the linear space of all functions with finite energy

$$\left(\iint_A (\nabla \Psi)^2 \, \mathrm{d}A < \infty \quad \text{if } \Psi(y, z) \in W_2^{(1)}(A) \right).$$

Notice that the finite energy requirement is *weaker* than the continuity of the Laplacian (see (3.1a)), needed in the classical formulation, and for this reason the following procedure is known by the name 'weak formulation' in the mathematical literature; see Ladyzhenskaya & Ural'tseva (1968) for further details.

To obtain the energy expression, the continuity equation (3.1a) is multiplied by an arbitrary $\Psi(y, z) \in W_2^{(1)}(A)$, integrated over A, and the Laplacian operator is then integrated by parts. With the help of the boundary conditions the following identity is obtained:

$$M(\phi, \Psi; K) = \iint_A \nabla \phi \cdot \nabla \Psi \, \mathrm{d}A + K^2 \iint_A \phi \Psi \, \mathrm{d}A - \Omega^2 \int_F \phi(y, 0)\, \Psi(y, 0) \, \mathrm{d}y + \sum_{n=0}^{\infty} \lambda_n(K)\, L_n^{\pm}(\phi)\, L_n^{\pm}(\Psi) \equiv 0, \tag{3.8}$$

where F is the free surface of A.

A *trapped mode* is a pair $\{K_T; T(y, z; K_T) \neq 0\}$ for which $M(T, \Psi; K_T) = 0$ for *all* $\Psi(y, z) \in W_2^{(1)}(A)$. If one observes that $M(\phi, \phi; K_T)$ is just twice the Lagrangian of the field $\phi(y, z)\, \mathrm{e}^{\mathrm{i}Kx}$, see (2.9), (3.6), (3.8), then the 'weak equation' $M(T, \Psi; K_T) = 0$ has a very clear physical meaning: it is just the stationary condition for the Lagrangian, as anticipated in §2.

Notice that the eigenvalue problem (3.8) can be defined in two alternative ways: in the first, Ω^2 is supposed given and the problem is to determine particular values of $K^2(\Omega)$; in the second, K^2 can be considered given and the problem is to determine particular values of $\Omega^2(K)$. The first alternative will be taken here, but identical results could as well be derived using the second. In both cases the coefficients $\{\lambda_n\}$ are nonlinear functions of the eigenvalues K^2 (or Ω^2) and so a special strategy is needed to prove the existence of a solution (and also to compute it numerically). A brief analogy with a discrete mechanical system can clarify the route to be followed. If a mechanical system has generalized mass and stiffness matrices, $\boldsymbol{b}(\Omega)$ and $\boldsymbol{c}(\Omega)$ respectively, both frequency dependent, the natural frequencies Ω_j can be determined in the following way: they are the roots of the equations $m_j(\Omega) = 0$, where $m_j(\Omega)$ are the eigenvalues of the matrix $\boldsymbol{M}(\Omega) = -\Omega^2 \boldsymbol{b}(\Omega) + \boldsymbol{c}(\Omega)$. In a similar way, let $\{m_j(K); E_j(y, z; K),\ j = 0, 1, 2, \ldots\}$ be the eigenvalues–eigenvectors of the bilinear form $M(.,.;K)$ or

$$M(E_j, \Psi; K) = m_j(K) \iint_A E_j \Psi \, \mathrm{d}A, \quad \text{all } \Psi(y, z) \in W_2^{(1)}(A). \tag{3.9}$$

Obviously, trapped modes are associated with the roots of the equations $m_j(K) = 0$, and attention is now turned to demonstrating the existence of at least one of these roots. But before this is done it is convenient to introduce some definitions that will be used later on. A definitive feature of a submerged body is the existence of a fluid region A_s above the body, where (see figure 1(*a*))

$$A_s = \{(y, z), |y| \leqslant b; 0 \leqslant z \leqslant -S\}.$$

If now $M_s(.,.; K)$ is the quadratic form

$$M_s(\Psi, \Psi; K) = \iint_{A_s} (\nabla\Psi)^2 \, dA_s + K^2 \iint_{A_s} \Psi^2 \, dA_s - \Omega^2 \int_F \Psi^2(y, 0) \, dy, \tag{3.10}$$

then obviously

$$M(\Psi, \Psi; K) \geqslant M_s(\Psi, \Psi; K); \quad \Psi(y, z) \in W_2^{(1)}(A). \tag{3.11}$$

The bilinear form $M(\phi, \Psi; K)$ is symmetric and the associated quadratic form $M(\phi, \phi; K)$ is bounded from below.† In this way the minimum (and also minimax) principle of eigenvalue theory can be used (see Courant & Hilbert 1953 for details) and it turns out that

$$\left.\begin{aligned} m_0(K) &= \min_{\text{all } \Psi} \frac{M(\Psi, \Psi; K)}{\iint_A \Psi^2 \, dA} = \frac{M(E_0, E_0; K)}{\iint_A E_0^2 \, dA}, \\ m_1(K) &= \min_{\Psi \perp E_0} \frac{M(\Psi, \Psi; K)}{\iint_A \Psi^2 \, dA} = \frac{M(E_1, E_1; K)}{\iint_A E_1^2 \, dA}, \end{aligned}\right\} \tag{3.12}$$

etc., where 'all' Ψ means an arbitrary element of $W_2^{(1)}(A)$, and $\Psi \perp E_0$ means that

$$\iint_A E_0 \Psi \, dA = 0.$$

If $\{K_0(\Omega); K_s(\Omega)\}$ are defined by the dispersion relations (2.1), (2.2) the following theorem can be easily proved:

THEOREM. *Let $M(.,.; K)$ be the bilinear form* (3.8) *and $\{m_j(K); j = 0, 1, 2, \ldots\}$ be the eigenvalues. Then*

(i) *$m_j(K)$, any j, is monotonically increasing with K;*

(ii) *for $K > K_s(\Omega)$, $m_j(K) > 0$; $j = 0, 1, 2, \ldots$;*

(iii) *for $K = K_0(\Omega)$ there exists at least one negative eigenvalue for all frequencies Ω ($m_0(K_0) < 0$ all $\Omega > 0$);*

(iv) *for $K = K_0(\Omega)$ there can be only a finite number of negative eigenvalues $\{m_0(K_0); m_1(K_0); \ldots; m_L(K_0)\}$, $L =$ finite;*

(v) *$m_0(K_0) \to 0^-$ when either $\Omega \to 0$ or $\Omega \to \infty$.*

Proof. Since $d\lambda_n/dK > 0$, see (3.5), then $M(\Psi, \Psi; K + \delta K) > M(\Psi, \Psi; K)$ if $\Psi \neq 0$ (but Ψ otherwise arbitrary). From the minimax principle of eigenvalue theory, the eigenvalues $m_j(K)$ must increase with K (see Courant & Hilbert 1953). This proves (i). To prove (ii) it suffices to show that the quadratic form $M_s(\Psi, \Psi; K)$, defined in (3.10), is positive definite for $K > K_s(\Omega)$. But in A_s the Fourier decomposition

$$\Psi(y, z) = \sum_{n=0}^{\infty} A_n(y) \bar{f}_n(z)$$

† $\int_F \phi^2(y, 0) \, dy \leqslant c(A)[\iint_A ((\nabla\phi)^2 + K_0(\Omega)\phi^2) \, dA]$, a result that is a direct consequence of the so-called Sobolev's embedding theorems; see Sobolev (1963).

can be used, where $\{\bar{f}_n(z)\}$ are defined by expressions like (3.2), with S in place of h. Using this expansion in $M_s(\Psi, \Psi; K)$ and the orthogonality of the modes $\{\bar{f}_n(z)\}$ it follows that

$$M_s(\Psi, \Psi; K) = (K^2 - K_s(\Omega)) \iint_{A_s} \Psi^2 \, dA.$$

So $M(\Psi, \Psi; K) > 0$ if $K > K_s(\Omega)$, and this proves (ii). From the minimum principle (3.12) it follows that

$$m_0(K_0) \leqslant \frac{M(f_0, f_0; K_0)}{\iint_A f_0^2 \, dA}, \tag{3.13}$$

where

$$\left.\begin{aligned} M(f_0, f_0; K_0) &= -I(K_0; \partial B), \\ I(K_0; \partial B) &= -\int_{\partial B} \frac{df_0}{dz} f_0 n_z \, d\,\partial B. \end{aligned}\right\} \tag{3.14}$$

In the above expression, n_z is the vertical component of the normal $\boldsymbol{n}$ to ∂B, where $\boldsymbol{n}$ points outward from the fluid region. The coefficient $I(K_0; \partial B)$ depends only on the wavenumber K_0 and the geometry ∂B of the cross-section, and it is an easy task to check that $I(K_0; \partial B)$ is always positive for a submerged body. So $m_0(K_0) < 0$, see (3.13) and (3.14), which proves (iii). Since the quadratic form $M(\Psi, \Psi; K_0)$ is bounded from below and the sequence $\{m_j(K_0)\}$ accumulates at infinity (see Courant & Hilbert 1953 for details), then $M(.,.; K_0)$ can have at most a finite number of negative eigenvalues. This proves (iv). When $\Omega \to \infty$, $\Omega^2 S$ also does, and $K_s(\Omega) \to \Omega^2 = K_0(\Omega)$. Then $m_0(K_s) \to m_0(K_0)$, but $m_0(K_s) > 0$ and $m_0(K_0) < 0$. The limit $m_0(K_s) \to m_0(K_0)$ is possible only if $m_0(K_0) \to 0^-$ when $\Omega \to \infty$. When $\Omega \to 0$,

$$M(\Psi, \Psi; K_0) \to \iint (\nabla \Psi)^2 \, dA + \sum_{n=1}^{\infty} \lambda_n(K_0) (L_n^{\pm}(\Psi))^2,$$

and this is a positive semidefinite quadratic form with minimum value equal to zero ($\Psi = 1$). Then $m_0(K_0) \to 0^-$ when $\Omega \to 0$ and this completes the demonstration of the theorem.

From (i), (ii) and (iii) an important conclusion can be derived: the equation $m_0(K) = 0$ has a unique solution $K_T(\Omega)$, where $K_0(\Omega) < K_T(\Omega) < K_s(\Omega)$. This proves the existence of at least one trapped mode. From (i) and (iv) it follows that the number of trapped modes is always finite. They correspond to the unique roots of the equations $m_j(K) = 0, j = 0, 1, \ldots, L$ and, from the minimum principle, these roots are ordered as

$$K_0(\Omega) < K_{T,L} \leqslant \ldots \leqslant K_{T,1} \leqslant K_{T,0} = K_T(\Omega).$$

From (v) it is easy to check that $K_T(\Omega)/K_0(\Omega) \to 1$ when $\Omega \to 0$ or $\Omega \to \infty$. These conclusions are summarized next.

COROLLARY. *Given an arbitrary submerged body*

(i) *for any frequency Ω there exists at least one trapped mode $\{K_T(\Omega); T_0(y, z; \Omega)\}$, where $K_T(\Omega)$ is the unique root of the equation $m_0(K) = 0$ and $T_0(y, z; \Omega) = E_0(y, z; K_T)$;*

(ii) $K_0(\Omega) < K_T(\Omega) < K_s(\Omega)$;

(iii) $K_T(\Omega)/K_0(\Omega) \to 1$ *when* $\Omega \to 0$ *or* $\Omega \to \infty$,

(iv) *for any frequency Ω there are at most $(L+1)$ trapped modes, $L =$ finite, where $K_{T,j}, j = 0, 1, \ldots, L$ are the unique roots of the equations $m_j(K) = 0, j = 0, 1, \ldots, L$ and $T_j(y, z; \Omega) = E_j(y, z; K_{T,j})$;*

(v) *if* $L \geqslant 1$ *then*

$$K_0(\Omega) < K_{T,L} \leqslant \ldots \leqslant K_{T,1} \leqslant K_{T,0} = K_T.$$

The trapped mode $\{K_T; T_0(y,z;\Omega)\}$ will be called the 'lowest trapped mode' since it is associated with the lowest eigenvalue $m_0(K)$. Since this is the only one whose existence is always granted it will be studied in some detail next. The scheme used to prove the existence of a trapped mode could also be used to determine it numerically. In particular, the Rayleigh quotient can be employed to determine a lower-bound approximation for $K_T(\Omega)$, a topic to be explored in the next section.

4. The lowest mode: some properties and an asymptotic approximation

Let $\{(K_T(\Omega); T_0(y,z;\Omega)\}$ be the lowest trapped mode at the frequency Ω. From (3.6) and (3.8) – see also (2.9) – it follows that

$$\iint_{A_\infty} \nabla T_0 \cdot \nabla \Psi \, dA_\infty + K_T^2 \iint_{A_\infty} T_0 \Psi \, dA = \Omega^2 \int_{-\infty}^{\infty} T_0(y,0)\, \Psi(y,0)\, dy. \tag{4.1}$$

Using (4.1) at the frequencies Ω and $\Omega + \delta\Omega$ and then letting $\delta\Omega \to 0$, one obtains

$$c = \frac{d\Omega}{dK_T} = \frac{K_T}{\Omega} \frac{\iint_{A_\infty} T_0^2(y,z)\, dA_\infty}{\int_{-\infty}^{\infty} T_0^2(y,0)\, dy}, \tag{4.2}$$

where, obviously, c is the longitudinal trapped-wave 'group velocity'. As should be expected, this value plays an important role in the nonlinear analysis to be addressed in the companion paper (Aranha 1988). Another relation that will be used then is

$$\frac{2K_T(\Omega)}{K_T(2\Omega)} < 1. \tag{4.3}$$

In fact, from (4.2),

$$\frac{dK_T}{d\Omega} = \frac{1}{c} = \frac{\Omega}{K_T} \frac{\int_{-\infty}^{\infty} T_0^2(y,0)\, dy}{\iint_{A_\infty} T_0^2(y,z)\, dA_\infty} = \frac{K_T}{\Omega} \frac{\Omega^2 \int_{-\infty}^{\infty} T_0^2(y,0)\, dy}{K_T^2 \iint_{A_\infty} T_0^2(y,z)\, dA_\infty} > \frac{K_T}{\Omega},$$

where the inequality is due to (4.1) with $\Psi = T_0$. Integrating now between Ω and 2Ω, relation (4.3) is obtained.

Once the existence of trapped modes is demonstrated it is important to know under which conditions this mode will be excited and, if so, what are the features of the response. It certainly helps to answer these questions if a convenient approximation for the lowest mode $\{K_T(\Omega); T_0(y,z;\Omega)\}$ is derived, and the remainder of this section is dedicated to this issue.

The basic idea is to use the Rayleigh quotient to approximate $K_T(\Omega)$, but a judicious choice of the trial function must be made. The following observation can be helpful in this context: since $I(K_0; \partial B) \to 0$ when $\Omega \to 0$ or $\Omega \to \infty$, see (3.14), then from item (iii) of the corollary (see §3) one obtains

$$\lim_{\substack{\Omega \to 0 \\ \Omega \to \infty}} T_0(y,z;\Omega) = f_0(z), \qquad \lim_{\substack{\Omega \to 0 \\ \Omega \to \infty}} \frac{K_T(\Omega)}{K_0(\Omega)} = 1.$$

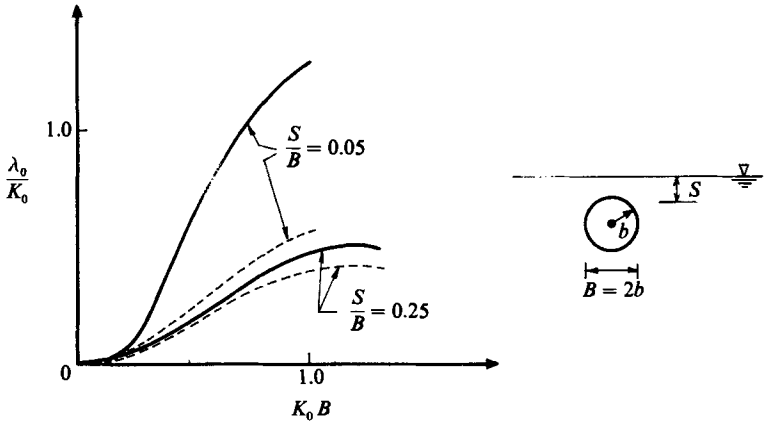


FIGURE 2. Comparison between the lower bound (4.5) (---) and the exact value computed by McIver & Evans (1985, figure 5*b*) (——).

The approximation $T_0(y, z; \Omega) \approx f_0(z)$ is then appropriate, for an arbitrary geometry, in the low or high frequency limits. Notice that *any submerged body* is transparent to the wave action in the limits $\Omega \to 0$ or $\Omega \to \infty$, and the parameter $I(K_0; \partial B)$ can gauge, in some sense, the intensity of the wave action. A structure relatively distant from the free surface is relatively transparent to wave action in the whole range of frequencies ($I(K_0; \partial B)$ is small for all Ω) and for this class of geometries the approximation

$$T_0(y, z; \Omega) \approx \frac{1}{\Omega} \begin{cases} f_0(z); & |y| \leqslant b, \\ e^{-\lambda_0(|y|-b)} f_0(z); & |y| \geqslant b, \end{cases} \tag{4.4}$$

should be appropriate† for an arbitrary Ω. Notice that the derivative of (4.4) is discontinuous at $y = b$, but this is permissible in the finite energy space $W_2^{(1)}(A)$ (in the same line, recall that trial functions used in the Rayleigh quotient must satisfy only the essential boundary conditions). Placing (4.4) into the identity $M(T_0, T_0; K_T) = 0$, the following quadratic equation in λ_0/K_0 is obtained:

$$\left.\begin{aligned} &\left(\frac{\lambda_0}{K_0}\right)^2 + C_1(\Omega)\left(\frac{\lambda_0}{K_0}\right) - C_2(\Omega) = 0, \\ &C_1(\Omega) = \frac{2}{K_0 \iint f_0^2 \, dA}; \quad C_2(\Omega) = \frac{C_1(\Omega)}{2K_0} I(K_0; \partial B) > 0. \end{aligned}\right\} \tag{4.5}$$

As has been said, the root of (4.5) gives a lower bound for λ_0/K_0, and it is asymptotically correct when $I(K_0; \partial B) \to 0$. In figure (2) this lower bound is compared with numerical results obtained from the full theory; see McIver & Evans (1985). As is clear, the approximation (4.5) improves drastically when S/B changes from 0.05 to 0.25.

Since the approximation (4.5) is better the smaller $I(K_0; \partial B)$ is, it is important to get an idea of the values of this parameter in the two cases shown in figure 2. For a circular cylinder in deep water it is not difficult to check that

$$I(K_0; \partial B) = 2\pi K_0^2 \, e^{-K_0^2(2S+B)} I_0(K_0); \; B = 1, \tag{4.6}$$

where $I_0(.)$ is the modified Bessel function (see Abramowitz & Stegun 1964).

† Indeed, for infinite water depth, (4.4) is correct in the limit $S \to \infty$, in which case $\lambda_0 \to 0$ and $T_0(y, z; \Omega) \to f_0(z)$.

From (4.6) it follows that, in the range $0 \leqslant K_0 B \leqslant 2$, $I(K_0; \partial B)$ is monotonically increasing with $K_0 B$ in both cases, its maximum values being 6.38 and 2.86 when S/B is equal to 0.05 and 0.25, respectively. The error, as shown in figure 2, also increases monotonically with $K_0 B$ in this range and for the deeper cylinder ($S/B = 0.25$) it is small for $K_0 B = 0.5$, when $I(K_0; \partial B) = 0.79$. A typical semisubmersible platform has two rectangular pontoons with beam $B \approx 16$ m, width $D \approx 8$ m, length $L \approx 100$ m, placed at depth $S \approx 12$ m and separated from each other by a distance $W \approx 60$ m (see Pinkster & Huijsmans 1982). Considering the two pontoons together in deep water it follows, in this case, that

$$I(K_0; \partial B) = 4K_0^2 \, \mathrm{e}^{-2K_0 S}(1-\mathrm{e}^{-2K_0 D}); \; B = 1. \tag{4.7}$$

The maximum of (4.7) is of order 0.74 for $K_0 B$ around 1.6. Since this value is comparable with the one for the deep circular cylinder ($S/B = 0.25$) at $K_0 B = 0.5$, it can be anticipated that approximation (4.5) should work very well for this class of ocean structures. The next section displays a direct application of expression (4.5).

5. The necessary condition for excitation of a trapped mode

A trapped mode can be excited if the incoming waves have the same frequency Ω and longitudinal wavenumber $K_T(\Omega)$ as the trapped mode. Since $K_T(\Omega) > K_0(\Omega)$, only nonlinear interactions of incoming waves can excite this mode. In this context, consider two incoming waves, one with frequency ω_1 and direction α_1 and the other with frequency $\omega_2 > \omega_1$ and direction α_2, where α_j is the angle between the wave direction and the x-axis.

In this way a trapped mode can be excited by a convenient combination of $\pm$ signs in expression (5.1), where

$$\sum_{i=1}^{2} \sum_{j=1}^{2} \exp[\pm \mathrm{i} K_0(\omega_j) \cos\alpha_j x \pm K_0(\omega_i) \cos\alpha_i x] \exp[\pm \mathrm{i}(\omega_j \pm \omega_i) t]$$

$$\equiv \exp[\mathrm{i}(K_T(\Omega) x - \Omega t)]. \tag{5.1}$$

Using the deep-water dispersion relation, $K_0(\omega_i) = \omega_i^2$, it is not difficult to check that only the combination $\omega_2 - \omega_1$ can excite a trapped mode, and then

$$\Omega = \omega_2 - \omega_1, \quad K_T(\Omega) = \omega_2^2 \cos\alpha_2 - \omega_1^2 \cos\alpha_1. \tag{5.2}$$

It is certainly possible to make a general analysis of this excitation condition, but a much clearer picture can be obtained if (5.2) is specialized to the class of geometries analysed in the last section, namely those that are relatively transparent to the wave action. In this case the parameter λ_0/K_0 is small, see (4.5), and from (3.5) it follows that

$$\frac{K_T(\Omega)}{K_0(\Omega)} = \left[1 + \left(\frac{\lambda_0}{K_0}\right)^2\right]^{\frac{1}{2}} \approx 1 + \frac{1}{2}\left(\frac{\lambda_0}{K_0}\right)^2. \tag{5.3}$$

Then with an error of order $0.5(\lambda_0/K_0)^2$ – roughly 2% for the semisubmersible analysed in the last section – $K_T(\Omega)$ can be taken equal to $K_0(\Omega) = \Omega^2$ in (5.2). In this way the simple excitation condition

$$\cos\alpha_2 = \frac{1 + r^2 \cos\alpha_1}{(1+r)^2}; \quad r = \frac{\omega_1}{\Omega} > 0, \tag{5.4}$$

can be obtained. So, for a given $\omega_1, \alpha_1, \omega_2$, there always exists an α_2 that satisfies (5.4)

and for which a trapped mode can be excited. Thus this phenomenon, although a rare event – as, in essence, all resonant phenomena are – is perfectly possible.

In nature the two harmonic waves considered could be associated with a local sea and a swell. If the waves are not pure harmonics but have a spectral density, given by the usual expression

$$\left.\begin{aligned} S_j(\omega) &= \frac{g^2}{\omega_j^5} f_j(\xi), \\ \xi &= \omega/\omega_j, \end{aligned}\right\} \quad (j = 1, 2) \tag{5.5}$$

where ω_j is the peak frequency and $f_j(.)$ is the spectral density function of form, then $(\xi\omega_1; \xi\omega_2)$ will also satisfy (5.4), for the same pair $(\alpha_1; \alpha_2)$, if $(\omega_1; \omega_2)$ do. In this case the excited trapped mode will have a frequency $\xi\Omega$ and so a spectrum of trapped modes will be excited. The important point is to observe that, in this circumstance, the whole spectrum of incoming waves will give energy to this resonant phenomenon.

6. Conclusion

In this paper the existence of trapped waves for arbitrary frequency and geometry of the submerged body has been demonstrated. Furthermore, an asymptotic approximation for the lowest trapped mode was derived and it has been shown that such an approximation should give good results for a body relatively transparent to wave action, such as a semisubmersible platform.

Necessary conditions for trapped mode excitation have been also analysed and it has been shown that, although a rare event in nature, it can be of importance since, when excited, all components of the sea spectrum give energy to this resonant phenomenon. In a companion paper (Aranha 1988) the features of the nonlinear resonant response will be analysed.

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